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Difference of composition operators between standard weighted Bergman spaces ☆

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ABSTRACT

We obtain estimates for the norm and essential norm of the difference of two composition operators between certain Bergman spaces. In particular, a necessary and sufficient condition for boundedness and compactness of the operator is established. Finally, we give a sufficient condition for boundedness and compactness of the difference operator between Hardy spaces.

1. Introduction

We denote by $H(\mathbb{D})$ the space of analytic functions on the open unit disk $\mathbb{D}$ in the complex plane. If φ is an analytic selfmap of $\mathbb{D}$ then it induces *a composition operator* $C_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$ defined by $C_\varphi f = f \circ \varphi$. If ψ is another analytic selfmap of $\mathbb{D}$, the pair (φ, ψ) induces *a difference operator* $C_\varphi - C_\psi$. One of the most important problems in the study of composition operators is to characterize compact differences of composition operators in Hardy spaces. This problem was posed in 1990 by J.H. Shapiro and C. Sundberg in [16]. The problem is related to the study of the topology of the set of composition operators. For more about these topics in Hardy and Bergman spaces, see [2,11–15]. Another important generalization of composition operators are *weighted composition operators*. That is, if u is a measurable function in $\mathbb{D}$, the weighted composition operator uC_φ is defined in $H(\mathbb{D})$ by $(uC_\varphi)f = u \cdot f \circ \varphi$. For more about weighted composition operators, see [1,3,4,12].

Throughout this paper there will be many statements that involve a pair of analytic selfmaps of $\mathbb{D}$. To make notations simpler these maps will always be denoted by φ and ψ and for such a pair we will denote $\sigma(z) := (\varphi(z) - \psi(z))/(1 - \overline{\varphi(z)}\psi(z))$ for every $z \in \mathbb{D}$.

In the present work we are mainly interested in boundedness and compactness of difference operators between (standard weighted) Bergman spaces A^p_α, $p > 0$, $\alpha > -1$. In [12] J. Moorhouse proved that the difference operator $C_\varphi - C_\psi$ is compact on A^2_α if and only if both

$$\lim_{|z|\to 1} \left|\sigma(z)\right| \frac{1-|z|^2}{1-|\varphi(z)|^2} = 0 \quad \text{and} \quad \lim_{|z|\to 1} \left|\sigma(z)\right| \frac{1-|z|^2}{1-|\psi(z)|^2} = 0.$$

In [8] some useful estimates were obtained for the essential norm of the difference operator using the ideas of Moorhouse. Recall, that the essential norm of an operator is its distance to the set of compact operators in operator norm.

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The motivation for our study is the following intuition. In [12] Moorhouse also proved that if u is a bounded analytic function on $\mathbb{D}$, then the weighted composition operator uC_φ is compact on A_α^2 if and only if

$$\lim_{|z|\to 1}\left|u(z)\right|\frac{1-|z|^2}{1-|\varphi(z)|^2}=0.$$

Thus Moorhouse's results suggest that there might be some connection between the difference operator $C_\varphi - C_\psi$ and the corresponding weighted composition operators σC_φ and σC_ψ. In this paper we show that this suggestion turns out to be true with interesting results.

If μ is a positive measure on $\mathbb{D}$ and $p>0$, denote $L^p(\mu)$ the Lebesgue space over the unit disk with respect to the measure μ. That is, $L^p(\mu)$ consists of all functions f defined in $\mathbb{D}$ for which

$$\|f\|_{L^p(\mu)} := \left[\int_{\mathbb{D}}\left|f(z)\right|^p\,d\mu\right]^{\frac{1}{p}} < \infty.$$

When $p \geqslant 1$, $\|\cdot\|_{L^p(\mu)}$ defines a norm and $L^p(\mu)$ becomes a Banach space.

If $\alpha > -1$ and A denotes the normalized Lebesgue area measure on $\mathbb{D}$, we will define the normed area measure A_α on $\mathbb{D}$ by

$$A_\alpha(E) := (\alpha+1)\int_E \left(1-|z|^2\right)^\alpha\,dA(z)$$

for every Lebesgue-measurable set $E \subset \mathbb{D}$. For simplicity, we will denote $L^p(A_\alpha) = L_\alpha^p$ and $\|\cdot\|_{L_\alpha^p} = \|\cdot\|_{\alpha,p}$. In terms of these notations, the Bergman space A_α^p is the subspace of L_α^p that consists of analytic functions in $\mathbb{D}$.

For all $p>0$ and $\alpha>-1$, define the *normed Bergman kernel function for* A_α^p by

$$k_a^{\alpha,p} := \left(\frac{1-|a|^2}{(1-\bar{a}z)^2}\right)^{\frac{\alpha+2}{p}}.$$

For the sake of notations, when dealing with the space A_α^p we will simply write $k_a = k_a^{\alpha,p}$.

For two real numbers A and B, the notation $A \lesssim B$ means that there exists a constant C such that $A \leqslant C\cdot B$. The constant C will be called a *comparability constant.* Furthermore, the notation $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$. In this case we say that A and B are *comparable*.

Suppose then that $p,q>0$, $\alpha,\beta>-1$ and T is a linear operator that maps A_α^p into Y^q, where Y^q is either A_β^q or $L^q(\mu)$. We will denote

$$\|T\|_{A_\alpha^p\to Y^q} := \sup\left\{\|Tf\|_{Y^q}:\ f\in A_\alpha^p,\ \|f\|_{\alpha,p}\leqslant 1\right\}$$

the operator norm of T. If $\|T\|_{A_\alpha^p\to Y^q}$ is finite, we say that T is a *bounded operator* from A_α^p into Y^q. The essential norm of the operator $T: A_\alpha^p \to Y^q$ will be denoted simply by $\|T\|_e$. That is,

$$\|T\|_e = \inf\left\{\|T-K\|_{A_\alpha^p\to Y^q}:\ K: A_\alpha^p\to Y^q \text{ compact}\right\}.$$

The main results of this paper are the following theorems.

Theorem A. *Suppose* $0<p\leqslant q<\infty$ *and* $\alpha,\beta>-1$. *Assume that functions* φ *and* ψ *are analytic selfmaps of* $\mathbb{D}$. *Then the operator* $C_\varphi - C_\psi$ *maps* A_α^p *into* A_β^q *if and only if both weighted composition operators* σC_φ *and* σC_ψ *map* A_α^p *into* L_β^q. *Furthermore,*

$$\|C_\varphi - C_\psi\|_{A_\alpha^p\to A_\beta^q} \approx \sup_{a\in\mathbb{D}}\left\|(C_\varphi - C_\psi)k_a\right\|_{\beta,q} \approx \max\left\{\|\sigma C_\varphi\|_{A_\alpha^p\to L_\beta^q},\ \|\sigma C_\psi\|_{A_\alpha^p\to L_\beta^q}\right\},$$

where the comparability constants will depend only on α, β, p *and* q.

Theorem B. *Suppose* $1<p\leqslant q<\infty$ *and* $\alpha,\beta>-1$. *Assume that* φ *and* ψ *are analytic selfmaps of* $\mathbb{D}$ *such that the operator* $C_\varphi - C_\psi : A_\alpha^p \to A_\beta^q$ *is bounded. Then*

$$\|C_\varphi - C_\psi\|_e \approx \limsup_{|a|\to 1}\left\|(C_\varphi - C_\psi)k_a\right\|_{\beta,q} \approx \max\left\{\|\sigma C_\varphi\|_e,\ \|\sigma C_\psi\|_e\right\},$$

where the operators σC_φ *and* σC_ψ *map* A_α^p *into* L_β^q *and the comparability constants will depend only on* α, β, p *and* q. *In particular,* $C_\varphi - C_\psi$ *is compact if and only if* $\limsup_{|a|\to 1}\|(C_\varphi - C_\psi)k_a\|_{\beta,q} = 0$.

For more about the essential norm in the case $p = 1$, see [6].

Recall that the *Hardy space* H^p, $p > 0$ is the set of functions analytic in $\mathbb{D}$ for which

$$\|f\|_{H^p} := \left[\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^p d\theta\right]^{\frac{1}{p}} < \infty.$$

Again, for $p \geqslant 1$ this defines a norm and H^p becomes a Banach space.

Similar techniques that are used to prove Theorems A and B can also be used to prove the following results concerning the difference operator between Hardy spaces. Below, H^p is considered as a subspace of $L^p(\partial\mathbb{D})$, the Lebesgue space over the boundary of the unit disk with standard arc-lenght measure. In addition, the function σ is considered as a function defined almost everywhere on the boundary of the unit disk (see Section 5).

Theorem C. *Suppose $1 < p \leqslant q < \infty$. Assume that φ and ψ are analytic selfmaps of $\mathbb{D}$ such that the operators σC_φ and σC_ψ map H^p into $L^q(\partial\mathbb{D})$. Then the difference operator $C_\varphi - C_\psi$ maps H^p into H^q. Furthermore, if σC_φ and σC_ψ are compact, then so is $C_\varphi - C_\psi$.*

Note that if a difference operator or the corresponding weighted composition operators map A^p_α into L^q_β (or H^p into $L^q(\partial\mathbb{D})$), then by the Closed Graph Theorem these operators are bounded.

2. Preliminaries

2.1. Pseudohyperbolic metric

Define the pseudohyperbolic metric $\rho : \mathbb{D} \times \mathbb{D} \to [0, 1)$ by

$$\rho(z, a) = \left|\frac{a - z}{1 - \bar{a}z}\right|.$$

The pseudohyperbolic metric obeys the following so-called strong form of triangle inequality:

$$\rho(z, w) \leqslant \frac{\rho(z, a) + \rho(a, w)}{1 + \rho(z, a)\rho(a, w)} \tag{2.1}$$

for all $a, z, w \in \mathbb{D}$. Furthermore, if $0 < r < 1$, then there exist constants A, B and C depending only on r such that whenever $z, w \in \mathbb{D}$ with $\rho(z, w) < r$,

$$A^{-1} \leqslant \frac{1 - |z|^2}{1 - |w|^2} \leqslant A, \tag{2.2}$$

$$B^{-1} \leqslant \left|\frac{1 - \bar{\xi}z}{1 - \bar{\xi}w}\right| \leqslant B \tag{2.3}$$

for all $\xi \in \mathbb{D}$ and

$$\frac{C^{-1}}{1 - |w|^2} \leqslant \frac{1 - |w|^2}{|1 - \overline{w}z|^2} \leqslant \frac{C}{1 - |w|^2}. \tag{2.4}$$

These estimates are elementary. Proofs can be found for example in [5], Section 2.5.

We will denote by $\triangle(a, r) := \{z \in \mathbb{D}\colon\ |\varphi_a(z)| < r\}$ the pseudohyperbolic disk centered at a with radius r.

2.2. Carleson measures

Definition 2.1. Let $0 < p \leqslant q < \infty$. A positive Borel measure μ on $\mathbb{D}$ is a *q-Carleson measure* for A^p_α if the inclusion map $I_\mu : A^p_\alpha \to L^q(\mu)$ is bounded i.e. there exists a constant M such that $\|f\|_{L^q(\mu)} \leqslant M\|f\|_{\alpha,p}$ for all $f \in A^p_\alpha$.

Next we will state the Carleson measure theorem for A^p_α. It has been obtained by several authors in different forms. The following is the most convenient for our purposes.

Theorem 2.2. *Let $0 < p \leqslant q < \infty$, $\alpha > -1$ and $0 < r < 1$. Suppose μ is a positive Borel measure on $\mathbb{D}$. Then the following are equivalent*:

(i) *The measure μ is a q-Carleson measure for A^p_α.*
(ii) $\|\mu\|_{\alpha,p,q,r} := \sup_{a\in\mathbb{D}} \frac{\mu(\triangle(a,r))}{(1-|a|^2)^{(2+\alpha)\frac{q}{p}}} < \infty$.
(iii) $\sup_{a\in\mathbb{D}} \|k_a\|^q_{L^q(\mu)} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (\frac{1-|a|^2}{|1-\bar{a}z|^2})^{(\alpha+2)\frac{q}{p}}\, d\mu < \infty$.

Furthermore, $\|I_\mu\|^q_{A^p_\alpha\to L^q(\mu)}$ and the quantities in (ii) *and* (iii) *are all comparable with comparability constants depending only on α, p, q and r.*

Proof. The equivalence of (i) and (ii) is proved in [9]. A method to prove the equivalence between (ii) and (iii) can be found for example in the proof of Theorem 7.4 in [18]. The comparability of the quantities follows from the proofs. □

Remark 2.3. We will usually apply Theorem 2.2 with $r = 1/2$. In this case we will simply write $\|\mu\|_{\alpha,p,q,\frac{1}{2}} = \|\mu\|_{\alpha,p,q}$.

2.3. Weighted composition operators

Suppose $u : \mathbb{D} \to \mathbb{C}$ is a measurable function and φ is an analytic selfmap of $\mathbb{D}$. Define measure $\mu^{q,\beta}_{u,\varphi}$ in $\mathbb{D}$ by

$$\mu^{q,\beta}_{u,\varphi}(E) = \int\limits_{\varphi^{-1}(E)} \left|u(z)\right|^q dA_\beta$$

for all Borel sets $E \subset \mathbb{D}$. By the measure theoretic change of variables $\|(uC_\varphi)f\|_{\beta,q} = \|f\|_{L^q(\mu^{q,\beta}_{u,\varphi})}$ for all $f \in H(\mathbb{D})$ (see [7], Section 39).

We will need the following results from [4]:

Theorem 2.4. *Suppose $0 < p \leqslant q < \infty$, $0 < r < 1$ and $\alpha, \beta > -1$. Let $u : \mathbb{D} \to \mathbb{C}$ be a measurable function and φ an analytic selfmap of $\mathbb{D}$. Then the following are equivalent*:

(i) *The weighted composition operator uC_φ maps A^p_α into L^q_β.*
(ii) $\|\mu^{q,\beta}_{u,\varphi}\|_{\alpha,p,q,r} < \infty$.
(iii) $\sup_{a\in\mathbb{D}} \|(uC_\varphi)k_a\|^q_{\beta,q} < \infty$.

Furthermore, $\|uC_\varphi\|^q_{A^p_\alpha\to L^q_\beta}$ and the quantities in (ii) *and* (iii) *are all comparable with comparability constants depending only on α, β, p, q and r.*

Proof. The claim of the proof follows directly from Theorem 2.2. □

Let $N \in \mathbb{N}$. Define the partial sum operator $S_N : H(\mathbb{D}) \to H(\mathbb{D})$ by

$$S_N\left(\sum_{k=0}^{\infty} a_k z^k\right) = \sum_{k=0}^{N} a_k z^k.$$

Define also $R_N = I - S_N$. It can be shown that these operators are uniformly bounded on A^p_α when $p > 1$ (see [17]). Furthermore, the operator S_N is clearly compact.

For $r \in (0,1)$ denote $\mathbb{D}_r = \{z \in \mathbb{D}\colon\ |z| < r\}$. If μ is a positive measure on $\mathbb{D}$, we will denote $\mu_r := \mu|(\mathbb{D}\setminus\mathbb{D}_r)$.

Theorem 2.5. *Suppose $1 < p \leqslant q < \infty$, $0 < r < 1$ and $\alpha, \beta > -1$. Let $u : \mathbb{D} \to \mathbb{C}$ be a measurable function and φ an analytic selfmap of $\mathbb{D}$ such that the operator $uC_\varphi : A^p_\alpha \to L^q_\beta$ is bounded. Then*

(i)

$$\|uC_\varphi\|^q_e \approx \lim_{s\to 1-} \left\|\left(\mu^{q,\beta}_{u,\varphi}\right)_s\right\|_{\alpha,p,q,r} \approx \liminf_{N\to\infty} \left\|(uC_\varphi)R_N\right\|^q_{A^p_\alpha\to L^q_\beta} \approx \limsup_{N\to\infty} \left\|(uC_\varphi)R_N\right\|^q_{A^p_\alpha\to L^q_\beta}$$
$$\approx \limsup_{|a|\to 1} \left\|(uC_\varphi)k_a\right\|^q_{\beta,q}$$

where the comparability constants depend only on α, β, p, q and r.

(ii) *For every* $0 < \eta < 1$,

$$\lim_{N\to\infty} \sup_{\|f\|_{\alpha,p}\leqslant 1} \int\limits_{\varphi^{-1}(\mathbb{D}_\eta)} \left|(uC_\varphi \circ R_N f)(z)\right|^q \, dA_\beta(z) = 0.$$

Proof. See Lemmas 1, 2 and the proof of Theorem 2 in [4]. Although in the proof of Theorem 2 it is assumed that the function u is analytic, the proof also works if it is only measurable (because the Carleson measure theorem works). $\square$

3. Upper bounds

In this section we will give upper estimates for the norm and the essential norm of the difference operator. The technique is similar to the one used in [8]. We will need the following important and known lemma.

Lemma 3.1. *Let* $0 < p \leqslant q < \infty$ *and* $\alpha > -1$. *There exists a constant* $C = C(\alpha, p, q)$ *such that*

$$\left|f(z) - f(a)\right|^q \leqslant C\rho(z,a)^q \frac{\int_{\triangle(a,\frac{1}{2})} |f(w)|^p \, dA_\alpha}{(1-|a|^2)^{(2+\alpha)q/p}}$$

for all $a \in \mathbb{D}$, $z \in \triangle(a, \frac{2-\sqrt{3}}{2})$ *and* $f \in A^p_\alpha$ *with* $\|f\|_{\alpha,p} \leqslant 1$.

Proof. For the case $p = q$, see [8] Lemma 3.5. The case $q > p$ is achieved by writing $|f(z) - f(a)|^q = (|f(z) - f(a)|^p)^{\frac{q}{p}}$ and using the fact that $\|f\|_{\alpha,p} \leqslant 1$. $\square$

Theorem 3.2. *Let* $0 < p \leqslant q < \infty$ *and* $\alpha, \beta > -1$. *Suppose* φ *and* ψ *are analytic selfmaps of* $\mathbb{D}$ *such that the operators* σC_φ *and* σC_ψ *map* A^p_α *into* L^q_β. *Then the following holds*:

(i) *The difference operator* $C_\varphi - C_\psi$ *maps* A^p_α *into* A^q_β *and*

$$\|C_\varphi - C_\psi\|^q_{A^p_\alpha \to A^q_\beta} \lesssim \max\left\{\left\|\mu^{q,\beta}_{\sigma,\varphi}\right\|_{\alpha,p,q}, \left\|\mu^{q,\beta}_{\sigma,\psi}\right\|_{\alpha,p,q}\right\},$$

where the comparability constant depends only on α, β, p *and* q.

(ii) *If* $p > 1$, *then*

$$\|C_\varphi - C_\psi\|^q_e \lesssim \max\left\{\lim_{r\to 1}\left\|\left(\mu^{q,\beta}_{\sigma,\varphi}\right)_r\right\|_{\alpha,p,q}, \lim_{r\to 1}\left\|\left(\mu^{q,\beta}_{\sigma,\psi}\right)_r\right\|_{\alpha,p,q}\right\}.$$

Proof. The technique to prove the first claim is quite similar to the proof of the second claim and it will be outlined after the proof of (ii). So, for a moment we suppose that (i) holds. Thus the difference operator is assumed to be bounded.

Since the partial sum operator S_N is compact for each $N \in \mathbb{N}$ we can estimate

$$\|C_\varphi - C_\psi\|_e \leqslant \limsup_{N\to\infty}\left\|(C_\varphi - C_\psi)R_N\right\|.$$

Denote $E = \{z \in \mathbb{D}\colon\ |\sigma(z)| \geqslant (2-\sqrt{3})/2\}$ and $E' = \mathbb{D} \setminus E$. Then

$$\begin{aligned} I_N(f) &:= \int\limits_E \left|(C_\varphi - C_\psi) \circ R_N f(z)\right|^q \, dA_\beta(z) \\ &\leqslant \left(\frac{4}{2-\sqrt{3}}\right)^q \left[\int\limits_E \left|(\sigma C_\varphi) \circ R_N f(z)\right|^q \, dA_\beta(z) + \int\limits_E \left|(\sigma C_\psi) \circ R_N f(z)\right|^q \, dA_\beta(z)\right] \\ &\leqslant \left(\frac{4}{2-\sqrt{3}}\right)^q \left[\left\|(\sigma C_\varphi)R_N\right\|^q + \left\|(\sigma C_\psi)R_N\right\|^q\right] \end{aligned}$$

whenever $\|f\|_{\alpha,p} \leqslant 1$, $N \in \mathbb{N}$. Thus by Theorem 2.5(i),

$$\limsup_{N\to\infty} \sup_{\|f\|_{\alpha,p}\leqslant 1} I_N(f) \lesssim \max\left\{\lim_{r\to 1}\left\|\left(\mu^{q,\beta}_{\sigma,\varphi}\right)_r\right\|_{\alpha,p,q}, \lim_{r\to 1}\left\|\left(\mu^{q,\beta}_{\sigma,\psi}\right)_r\right\|_{\alpha,p,q}\right\}.$$

Denote

$$J_N(f) := \int\limits_{E'} \left|(C_\varphi - C_\psi) \circ R_N f(z)\right|^q \, dA_\beta(z).$$

Let $0 < r < 1$ be arbitrary. Suppose $z \in E' \cap \varphi^{-1}(\mathbb{D}_r)$. By the strong form of triangle inequality of the pseudohyperbolic metric, we can find $r' \in (0, 1)$ such that $E' \cap \varphi^{-1}(\mathbb{D}_r) \subset \psi^{-1}(\mathbb{D}_{r'})$. Thus by Theorem 2.5(ii),

$$\lim_{N\to\infty} \sup_{\|f\|_{\alpha,p}\leqslant 1} \int_{E'\cap\varphi^{-1}(\mathbb{D}_r)} \left|(C_\varphi \circ R_N f)(z)\right|^q dA_\beta(z) = 0$$

and

$$\lim_{N\to\infty} \sup_{\|f\|_{\alpha,p}\leqslant 1} \int_{E'\cap\varphi^{-1}(\mathbb{D}_r)} \left|(C_\psi \circ R_N f)(z)\right|^q dA_\beta(z) = 0.$$

Hence

$$\limsup_{N\to\infty} \sup_{\|f\|_{\alpha,p}\leqslant 1} J_N(f) \lesssim \sup_{\|f\|_{\alpha,p}\leqslant 1} \int_F \left|(C_\varphi - C_\psi) f(z)\right|^q dA_\beta(z),$$

where $F = E' \cap \varphi^{-1}(\mathbb{D} \setminus \mathbb{D}_r)$. In the estimate above we also used the fact that the operators R_N are uniformly bounded. Using Lemma 3.1, Fubini's theorem and inequality (2.2) we can estimate

$$\begin{aligned}
\int_F \left|(C_\varphi - C_\psi) f(z)\right|^q dA_\beta(z) &\lesssim \int_F \left|\sigma(z)\right|^q \frac{\int_{\triangle(\varphi(z),\frac{1}{2})} |f(w)|^p\, dA_\alpha(w)}{(1-|\varphi(z)|^2)^{(2+\alpha)\frac{q}{p}}}\, dA_\beta(z) \\
&\lesssim \int_{\mathbb{D}} \left|f(w)\right|^p \frac{\int_{\varphi^{-1}(\triangle(w,\frac{1}{2}))\cap F} |\sigma(z)|^q\, dA_\beta(z)}{(1-|w|^2)^{(\alpha+2)\frac{q}{p}}}\, dA_\alpha(w) \\
&\leqslant \int_{\mathbb{D}} \left|f(w)\right|^p \frac{\int_{\varphi^{-1}(\triangle(w,\frac{1}{2})\cap\mathbb{D}\setminus\mathbb{D}_r)} |\sigma(z)|^q\, dA_\beta(z)}{(1-|w|^2)^{(\alpha+2)\frac{q}{p}}}\, dA_\alpha(w) \\
&\leqslant \|f\|_{\alpha,p}^p \left\|\left(\mu_{\sigma,\varphi}^{q,\beta}\right)_r\right\|_{\alpha,p,q}.
\end{aligned}$$

Letting $r \to 1$ and using the above estimates we get

$$\begin{aligned}
\|C_\varphi - C_\psi\|_e^q &\leqslant \limsup_{N\to\infty} \left\|(C_\varphi - C_\psi) R_N\right\|^q \\
&\leqslant \limsup_{N\to\infty} \sup_{\|f\|_{\alpha,p}\leqslant 1} I_N(f) + \limsup_{N\to\infty} \sup_{\|f\|_{\alpha,p}\leqslant 1} J_N(f) \\
&\lesssim \max\left\{ \lim_{r\to 1} \left\|\left(\mu_{\sigma,\varphi}^{q,\beta}\right)_r\right\|_{\alpha,p,q}, \lim_{r\to 1} \left\|\left(\mu_{\sigma,\psi}^{q,\beta}\right)_r\right\|_{\alpha,p,q} \right\}.
\end{aligned}$$

This proves (ii).

To prove (i), take $f \in A_\alpha^p$ such that $\|f\|_{\alpha,p} \leqslant 1$. Divide the proof in two parts as above using the pseudohyperbolic distance between φ and ψ. The case when $|\sigma|$ is bounded away from zero is treated the same way as above. When $|\sigma|$ is bounded away from 1, apply Lemma 3.1 and Theorem 2.4. □

4. Lower bounds

Next we will produce a lower bound for the essential norm. For that we will need a few easy lemmas. The first one is well known.

Lemma 4.1. *Let X and Y be Banach spaces and $T : X \to Y$ a bounded linear operator. Let $(k_n)_{n=1}^\infty$ be any sequence in X such that $\|k_n\|_X = 1$ for every n and $k_n \to 0$ weakly when $n \to \infty$. Then*

$$\|T\|_e \geqslant \limsup_{n\to\infty} \left\|T(k_n)\right\|_Y.$$

Remark 4.2. Suppose (a_n) is a sequence in $\mathbb{D}$ such that $|a_n| \to 1$ as $n \to \infty$. Then the functions $k_{a_n}(z) = \left(\frac{1-|a_n|^2}{(1-\overline{a_n}z)^2}\right)^{\frac{\alpha+2}{p}}$ form a suitable test sequence in the space A_α^p in the sense of the previous lemma.

Lemma 4.3. *Suppose* $0 < r < 1$. *There exists a constant* $C = C(r) > 0$ *such that whenever* $a \in \mathbb{D}$ *and* $z \in \triangle(a, r)$,

$$\frac{|a|}{C}\rho(z,w) \leqslant \left|1 - \frac{1-\bar{a}z}{1-\bar{a}w}\right| \leqslant |a|C\rho(z,w)$$

for every $w \in \mathbb{D}$.

Proof. Let $a, z \in \mathbb{D}$ such that $\rho(a,z) < r$. For every $w \in \mathbb{D}$ we can write

$$\left|1 - \frac{1-\bar{a}z}{1-\bar{a}w}\right| = |a|\left|\frac{z-w}{1-\bar{z}w}\right|\left|\frac{1-\bar{z}w}{1-\bar{a}w}\right|.$$

Now the desired estimate follows from (2.3). □

Lemma 4.4. *Suppose* $0 < r < 1$, $\gamma > 0$. *Then there exists a constant* $C = C(r,\gamma)$ *such that whenever* $a \in \mathbb{D}$ *and* $z \in \triangle(a, r)$,

$$\left|\left(\frac{1-|a|^2}{(1-\bar{a}z)^2}\right)^{\gamma} - \left(\frac{1-|a|^2}{(1-\bar{a}w)^2}\right)^{\gamma}\right| \geqslant C\frac{|a|\rho(z,w)}{(1-|a|^2)^{\gamma}}$$

for all $w \in \mathbb{D}$.

Proof. Let $a, w \in \mathbb{D}$ and $z \in \triangle(a, r)$ be arbitrary. Write

$$\left|\left(\frac{1-|a|^2}{(1-\bar{a}z)^2}\right)^{\gamma} - \left(\frac{1-|a|^2}{(1-\bar{a}w)^2}\right)^{\gamma}\right| = \left|\frac{1-|a|^2}{(1-\bar{a}z)^2}\right|^{\gamma} \cdot \left|1 - \left(\frac{1-\bar{a}z}{1-\bar{a}w}\right)^{2\gamma}\right|. \tag{4.1}$$

We will first estimate the second coefficient on the right. Write $\xi = (\frac{1-\bar{a}z}{1-\bar{a}w})^{2\gamma}$ and $\lambda = \frac{1}{2\gamma}$. By Lemma 4.3 and the triangle inequality, $|\xi| \leqslant R$ for some $R \geqslant 1$ depending only on r and γ. If $\operatorname{Re}\xi \geqslant \frac{1}{2}$, then by the mean value inequality for complex functions,

$$\left|1 - \xi^{\lambda}\right| \leqslant \lambda \max\{2^{1-\lambda}, R^{\lambda-1}\}|1-\xi|.$$

On the other hand, if $\operatorname{Re}\xi < \frac{1}{2}$, then $|1-\xi| > \frac{1}{2}$ so that

$$\left|1 - \xi^{\lambda}\right| \leqslant 2\left(1 + |\xi|^{\lambda}\right)|1-\xi| \leqslant 2\left(1 + R^{\lambda}\right)|1-\xi|.$$

Thus there exists a constant $C > 0$ depending on r and γ such that

$$\left|1 - \left(\frac{1-\bar{a}z}{1-\bar{a}w}\right)^{2\gamma}\right| = |1-\xi| \geqslant C\left|1-\xi^{\lambda}\right| = C\left|1 - \left(\frac{1-\bar{a}z}{1-\bar{a}w}\right)\right|.$$

Applying this, Lemma 4.3 and inequality (2.4) to Eq. (4.1) yield the claim. □

Theorem 4.5. *Let* $0 < p \leqslant q < \infty$ *and* $\alpha, \beta > -1$. *Suppose* φ *and* ψ *are analytic selfmaps of* $\mathbb{D}$ *such that the operator* $C_\varphi - C_\psi$ *maps* A^p_α *into* A^q_β. *Then*

(i) *the operators* σC_φ *and* σC_ψ *map* A^p_α *into* L^q_β *and*

$$\sup_{a\in\mathbb{D}}\left\|(C_\varphi - C_\psi)k_a\right\|^q_{\beta,q} \gtrsim \max\left\{\left\|\mu^{q,\beta}_{\sigma,\varphi}\right\|_{\alpha,p,q}, \left\|\mu^{q,\beta}_{\sigma,\psi}\right\|_{\alpha,p,q}\right\};$$

(ii)

$$\limsup_{|a|\to 1}\left\|(C_\varphi - C_\psi)k_a\right\|^q_{\beta,q} \gtrsim \max\left\{\lim_{r\to 1}\left\|\left(\mu^{q,\beta}_{\sigma,\varphi}\right)_r\right\|_{\alpha,p,q}, \lim_{r\to 1}\left\|\left(\mu^{q,\beta}_{\sigma,\psi}\right)_r\right\|_{\alpha,p,q}\right\}.$$

Above, the comparability constants depend only on α, p *and* q.

Proof. Again, the proof of (i) will be outlined after proving (ii). For that it is enough to show that

$$\Gamma := \limsup_{|a|\to 1}\left\|(C_\varphi - C_\psi)k_a\right\|^q \gtrsim \lim_{r\to 1}\left\|\left(\mu^{q,\beta}_{\sigma,\varphi}\right)_r\right\|_{\alpha,p,q},$$

because after that the roles of φ and ψ can be changed to get the other lower bound. Applying the previous lemma we get

$$
\begin{aligned}
\Gamma &= \limsup_{|a|\to 1} \int_{\mathbb{D}} \left| \left(\frac{1-|a|^2}{(1-\bar{a}\varphi(z))^2} \right)^{\frac{\alpha+2}{p}} - \left(\frac{1-|a|^2}{(1-\bar{a}\psi(z))^2} \right)^{\frac{\alpha+2}{p}} \right|^q dA_\beta(z) \\
&\gtrsim \limsup_{|a|\to 1} \int_{\varphi^{-1}(\triangle(a,\frac{1}{2}))} \frac{|\sigma(z)|^q}{(1-|a|^2)^{(\alpha+2)\frac{q}{p}}}\, dA_\beta(z) \\
&= \lim_{r\to 1} \sup_{|a|>\frac{r-\frac{1}{2}}{1-\frac{r}{2}}} \frac{\mu_{\sigma,\varphi}^{q,\beta}(\triangle(a,\frac{1}{2}))}{(1-|a|^2)^{(\alpha+2)\frac{q}{p}}} \\
&\geqslant \lim_{r\to 1} \sup_{|a|>\frac{r-\frac{1}{2}}{1-\frac{r}{2}}} \frac{\mu_{\sigma,\varphi}^{q,\beta}(\triangle(a,\frac{1}{2}) \cap (\mathbb{D}\setminus\mathbb{D}_r))}{(1-|a|^2)^{(\alpha+2)\frac{q}{p}}} \\
&= \lim_{r\to 1} \big\| \big(\mu_{\sigma,\varphi}^{q,\beta}\big)_r \big\|_{\alpha,p,q},
\end{aligned}
$$

since $\triangle(a,\frac{1}{2}) \cap (\mathbb{D}\setminus\mathbb{D}_r) \neq \emptyset$ if and only if $|a| \geqslant \frac{r-\frac{1}{2}}{1-\frac{r}{2}}$.

To prove (i) apply Lemma 4.4 as above to obtain

$$
\sup_{|a|\geqslant 2^{-3}} \frac{\mu_{\sigma,\varphi}^{q,\beta}(\triangle(a,\frac{1}{2}))}{(1-|a|^2)^{(\alpha+2)\frac{q}{p}}} \lesssim \sup_{|a|\geqslant 2^{-3}} \big\| (C_\varphi - C_\psi) k_a \big\|^q.
$$

Moreover, notice that $\triangle(a,2^{-3}) \subset \triangle(2^{-2},2^{-1})$ for every $a \in \triangle(0,2^{-3})$. Therefore

$$
\sup_{|a|<2^{-3}} \frac{\mu_{\sigma,\varphi}^{q,\beta}(\triangle(a,2^{-3}))}{(1-|a|^2)^{(\alpha+2)\frac{q}{p}}} \leqslant \frac{\mu_{\sigma,\varphi}^{q,\beta}(\triangle(2^{-2},\frac{1}{2}))}{(1-(2^{-2})^2)^{(\alpha+2)\frac{q}{p}}}.
$$

Now the above inequalities yield

$$
\sup_{a\in\mathbb{D}} \frac{\mu_{\sigma,\varphi}^{q,\beta}(\triangle(a,2^{-3}))}{(1-|a|^2)^{(\alpha+2)\frac{q}{p}}} \lesssim \sup_{a\in\mathbb{D}} \big\| (C_\varphi - C_\psi) k_a \big\|^q.
$$

The claim (i) now follows by Theorem 2.2. $\square$

It is now an easy task to proof the main theorems.

Proof of Theorem A. Apply Theorems 3.2(i) and 4.5(i) to obtain

$$
\sup_{a\in\mathbb{D}} \big\| (C_\varphi - C_\psi) k_a \big\|_{\beta,q}^q \leqslant \| C_\varphi - C_\psi \|_{A_\alpha^p \to A_\beta^q}^q \lesssim \max\big\{ \big\| \mu_{\sigma,\varphi}^{q,\beta} \big\|_{\alpha,p,q}, \big\| \mu_{\sigma,\psi}^{q,\beta} \big\|_{\alpha,p,q} \big\} \lesssim \sup_{a\in\mathbb{D}} \big\| (C_\varphi - C_\psi) k_a \big\|_{\beta,q}^q,
$$

where the comparability constants depend only on α, β, p and q. The comparability of the norm of the difference operator and the norms of the corresponding weighted composition operators follows from the above inequalities and Theorem 2.4. $\square$

Proof of Theorem B. Applying Lemma 4.1, Remark 4.2, Theorem 3.2(ii) and Theorem 4.5(ii) we obtain

$$
\begin{aligned}
\limsup_{|a|\to 1} \big\| (C_\varphi - C_\psi) k_a \big\|_{\beta,q}^q &\leqslant \| C_\varphi - C_\psi \|_e^q \\
&\lesssim \max\Big\{ \lim_{r\to 1} \big\| \big(\mu_{\sigma,\varphi}^{q,\beta}\big)_r \big\|_{\alpha,p,q}, \lim_{r\to 1} \big\| \big(\mu_{\sigma,\psi}^{q,\beta}\big)_r \big\|_{\alpha,p,q} \Big\} \\
&\lesssim \limsup_{|a|\to 1} \big\| (C_\varphi - C_\psi) k_a \big\|_{\beta,q}^q,
\end{aligned}
$$

where the comparability constants depend only on α, β, p and q. The comparability of the essential norm of the difference operator and the essential norms of the corresponding weighted composition operators follows from the above inequalities and Theorem 2.5. $\square$

5. Sufficient condition between Hardy spaces

It is a classical result that if $f \in H^p$ for some $p > 0$, then it has a radial extension to the boundary of the disk almost everywhere i.e. there exists a function f^* defined on closed unit disk such that $f^*(z) = f(z)$ for every $z \in \mathbb{D}$ and $f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ for almost every $\theta \in [0, 2\pi)$. Furthermore,

$$\|f\|_{H^p}^p = \frac{1}{2\pi}\int_0^{2\pi} \left|f^*\left(e^{i\theta}\right)\right|^p d\theta.$$

Note that the function σ also has a well-defined radial extension σ^* almost everywhere on the boundary. Indeed, if $\varphi \neq \psi$, then by a classical result the radial limits of φ and ψ can coincide only on a set of measure zero.

Below, H^p is considered via the radial extension as a subspace of $L^p(\partial\mathbb{D})$. We will identify the functions in H^p and their radial extensions. We will also use the same notation for the function σ and its radial extension.

Denote Pf the Poisson transformation of a function $f \in L^1(\partial\mathbb{D})$. Recall that

$$Pf(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|1-\bar{z}e^{i\theta}|^2} f\left(e^{i\theta}\right) d\theta$$

for every $z \in \mathbb{D}$.

Lemma 5.1. *Let $f \in H^1$ and $0 < r < 1$ be arbitrary. Then there exists a constant $C = C(r) > 0$ such that*

$$\left|f(z) - f(a)\right| \leqslant C\rho(z,a)P|f|(a)$$

for all $z \in \triangle(a, r)$.

Proof. The claim follows from the corresponding property for the Poisson kernel. □

In the following, if $A \subset \mathbb{C}$, then $\overline{A}$ denotes the closure of A with respect to the Euclidean metric in $\mathbb{C}$.

Definition 5.2. Suppose $0 < p \leqslant q < \infty$. Let μ be a positive Borel measure in $\overline{\mathbb{D}}$. Then the measure μ is called *a q-Carleson measure for H^p* if there exists a constant $M > 0$ such that

$$\left(\int_{\overline{\mathbb{D}}} \left|f(z)\right|^q d\mu(z)\right)^{\frac{1}{q}} \leqslant M\|f\|_{H^p}$$

for all $f \in H^p$. Furthermore, μ is called *a vanishing q-Carleson measure* if the inclusion map $I_\mu : H^p \to L^q(\mu)$ is compact.

Like in the case of Bergman spaces, we can characterize bounded and compact weighted composition operators by studying the properties of certain measures.

Theorem 5.3. *Suppose $0 < p \leqslant q < \infty$. Let u be a measurable function in $\partial\mathbb{D}$. Furthermore, let φ be an analytic selfmap of $\mathbb{D}$. Define the measure $\mu_{u,\varphi}$ by*

$$\mu_{u,\varphi}(E) = \int_{\varphi^{-1}(E)\cap\partial\mathbb{D}} \left|u(z)\right|^q dm$$

for every Borel set $E \subset \overline{\mathbb{D}}$, where dm is the normalized Lebesgue measure over the boundary of the unit disk. Then

(a) *the operator uC_φ maps H^p into $L^q(\partial\mathbb{D})$ if and only if $\mu_{u,\varphi}$ is a q-Carleson measure*;
(b) *the operator $uC_\varphi : H^p \to L^q(\partial\mathbb{D})$ is compact if and only if $\mu_{u,\varphi}$ is a vanishing q-Carleson measure.*

Proof. The claims follow from the fact that $\|f\|_{L^q(\mu_{u,\varphi})} = \|(uC_\varphi)f\|_{L^q(\partial\mathbb{D})}$ (see [1], Lemma 2.1 for details). □

Remark 5.4. It is a classical result that in the case $p > 1$ we can replace H^p in Definition 5.2 and in Theorem 5.3 with *the harmonic Hardy space h^p* consisting of Poisson transformations of $L^p(\partial\mathbb{D})$-functions.

Proof of Theorem C. The boundedness case is omitted as it is almost identical to the compactness case.

Let f_n be a sequence in H^p such that $f_n \to 0$ weakly when $n \to \infty$. Define $E := \{\theta \in [0, 2\pi)\colon\ \varphi(e^{i\theta}) \text{ and } \psi(e^{i\theta}) \text{ exist}\}$. Then E^c has measure zero. In addition, define $F := \{\theta \in E\colon\ \varphi(e^{i\theta}) \neq \psi(e^{i\theta})\}$ so that F^c has measure zero when $\varphi \neq \psi$. Dividing the proof in two parts as in the proof of Theorem 3.2 it is enough to show that

$$\int\limits_{\{\theta\in F\colon |\sigma(e^{i\theta})|<\frac{1}{2}\}} \left|f_n \circ \varphi\left(e^{i\theta}\right) - f_n \circ \psi\left(e^{i\theta}\right)\right|^q d\theta \to 0, \quad \text{as } n \to \infty.$$

If $|\varphi(e^{i\theta})| = 1$ for some $\theta \in F$, then $|\sigma(e^{i\theta})| = 1$. Thus

$$\left\{\theta \in F\colon \left|\sigma\left(e^{i\theta}\right)\right| < 1/2\right\} \subset \left\{\theta \in F\colon \left|\varphi\left(e^{i\theta}\right)\right| < 1 \text{ and } \left|\psi\left(e^{i\theta}\right)\right| < 1\right\}.$$

Now using Lemma 5.1 we get

$$\begin{aligned}\int\limits_{\{\theta\in F\colon |\sigma(e^{i\theta})|<\frac{1}{2}\}} \left|f_n \circ \varphi\left(e^{i\theta}\right) - f_n \circ \psi\left(e^{i\theta}\right)\right|^q d\theta &\lesssim \int\limits_{\{\theta\in F\colon |\sigma(e^{i\theta})|<\frac{1}{2}\}} \left|\sigma\left(e^{i\theta}\right)\right|^q \left(P|f_n| \circ \varphi\left(e^{i\theta}\right)\right)^q d\theta \\ &\leqslant \int\limits_{\overline{\mathbb{D}}} \left(P|f_n|(z)\right)^q d\mu_{\sigma,\varphi} \to 0\end{aligned}$$

by Remark 5.4 and the fact that the operator σC_φ is compact. Thus the difference operator is also compact. □

6. Final remarks

The proofs in this paper are built using pseudohyperbolic discs of radius 1/2. Actually we could take any radius r between 0 and 1. The proofs would still work although the constants in various inequalities would change according to r.

Theorems A and B motivate the study of a possible connection between the difference operator from A^p_α into A^q_β, where $p > q$, and the corresponding weighted composition operators. By Pitt's theorem and the fact that the space A^p_α, $p > 1$, $\alpha > -1$ is isomorphic to the sequence space l^p, the operators in question are compact if and only if it they are bounded. Despite this helping result the problem does not seem so easy. One reason for difficulties is the fact that in the case $p > q$ the Carleson measures are much harder to describe (see [10]).

Finally, Theorem C gives rise to the following question: Is boundedness and compactness of the difference operator in Hardy spaces enough to guarantee boundedness and compactness of the corresponding weighted composition operators? An answer to this question might bring some light to the problem of characterizing the compact differences of composition operators in Hardy spaces.

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